PDE of Neuron Models

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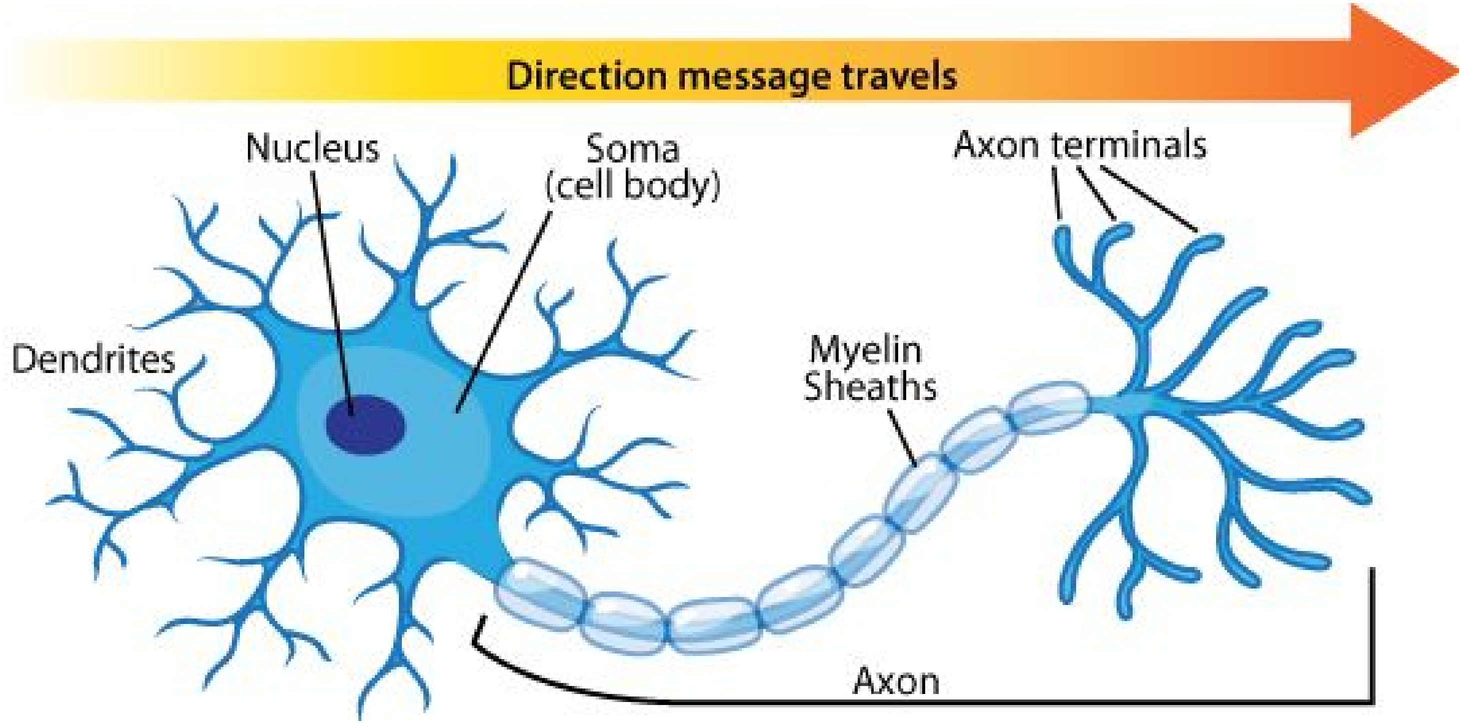
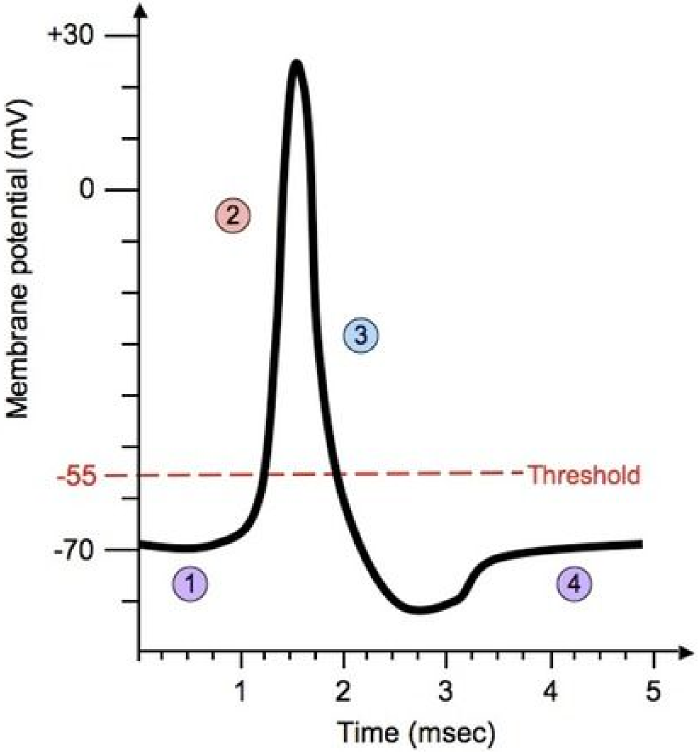
This is a 10 minute introduction to an introduction.

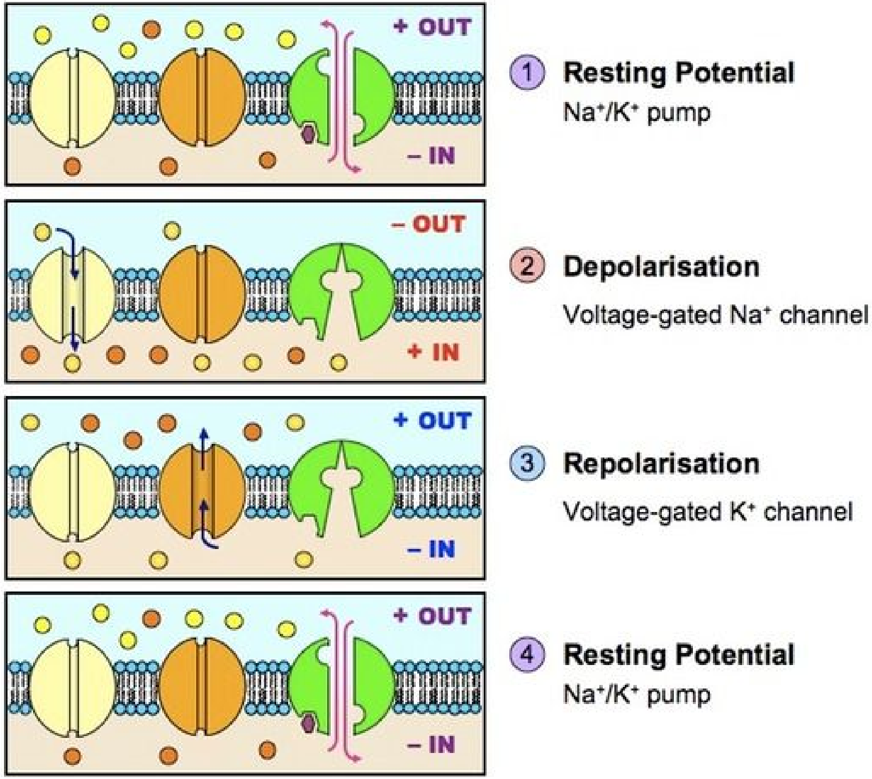
1. So what's a neuron, again?
2. Action Potential
3. Hodgkin-Huxley
4. FitzHugh-Nagumo
   a. 'Fast' and 'Slow' Time
   b. Phase Space
   c. Space *and* Time: Travelling Waves

This is a neuron:

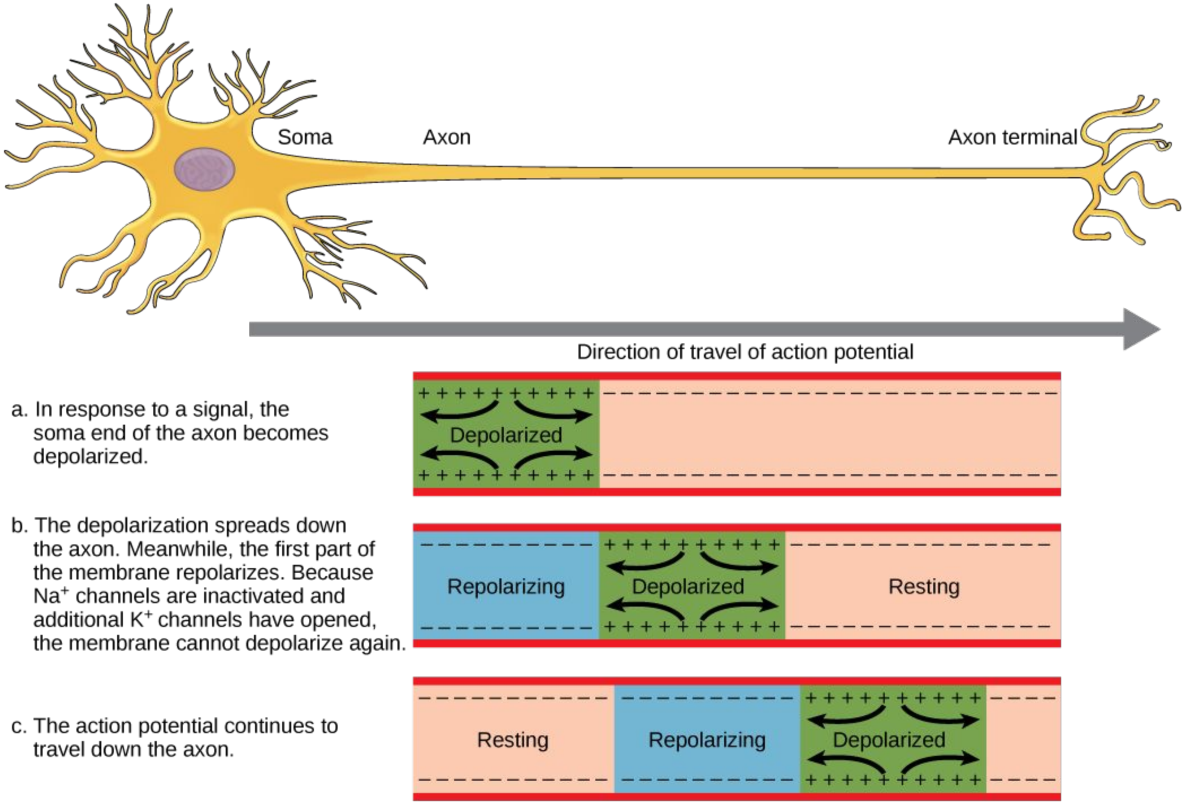


Action Potential





Action Potential



Please Don't Panic: Hodgkin-Huxley

By applying Q=CV to the Ion Cascades:

Current/Area = sum of currents

$$\begin{cases} I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_\mathrm{K} n^4 (V_m - V_K) + \bar{g}_\mathrm{Na} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l) \\ \frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n \\ \frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m \\ \frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h \end{cases}$$

Terms are each the constituent currents/area

Hodgkin-Huxley

Max Capacitance

Membrane Voltage

$$\begin{cases} I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_\mathrm{K} n^4 (V_m - V_K) + \bar{g}_\mathrm{Na} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l) \\ \frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n \\ \frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m \\ \frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h \end{cases}$$

Variable [0,1] characterizing activation

Channel Voltage Limit

Hodgkin-Huxley

$$\begin{cases} I = C_m \frac{\mathrm{d}V_m}{\mathrm{d}t} + \bar{g}_{\mathrm{K}} n^4 (V_m - V_K) + \bar{g}_{\mathrm{Na}} m^3 h (V_m - V_{Na}) + \bar{g}_l (V_m - V_l) \\ \frac{dn}{dt} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n \\ \frac{dm}{dt} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m \\ \frac{dh}{dt} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h \end{cases}$$

Sodium Channel Activation

Potassium
Channel Activation

Sodium Channel
Deactivation

FitzHugh-Nagumo

- Frustration ➔ Simpler model based on the van der Pol oscillator:

Nonlinear competing term feedback

Voltage variable

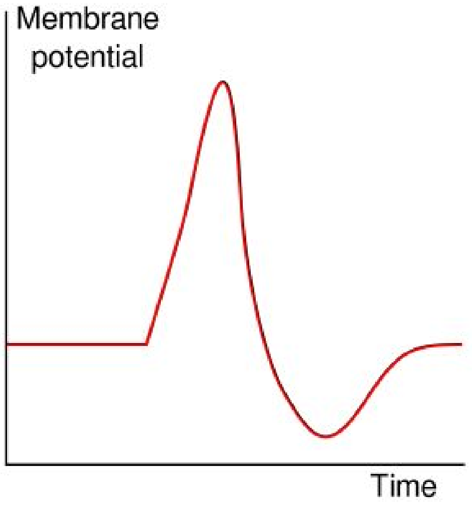
"Recovery" variable

$$\begin{cases} \dot{V} &= V - \frac{V^3}{3} - W + I \\ \dot{W} &= 0.08(V + 0.7 - 0.8W) \end{cases}$$

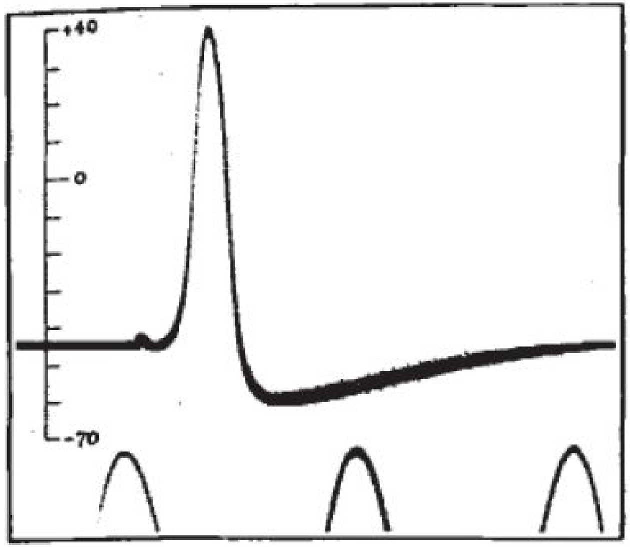
Linear, negative, and smaller magnitude feedback

- 2 state variables ➔ much easier to visualize and analyze

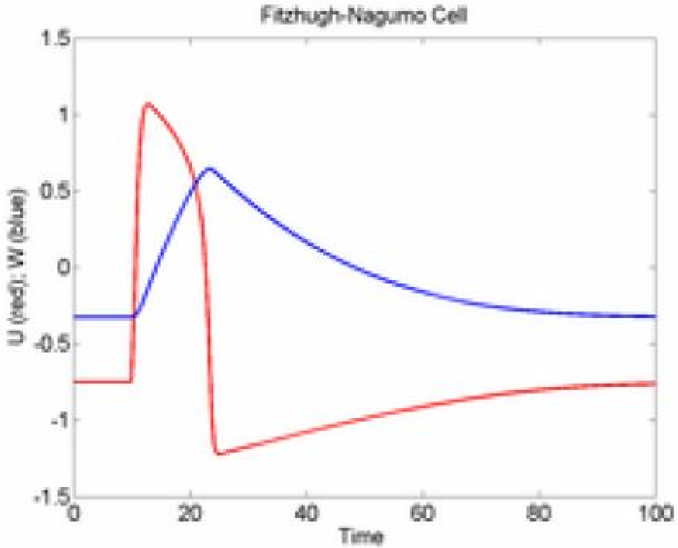
Outputs



Actual



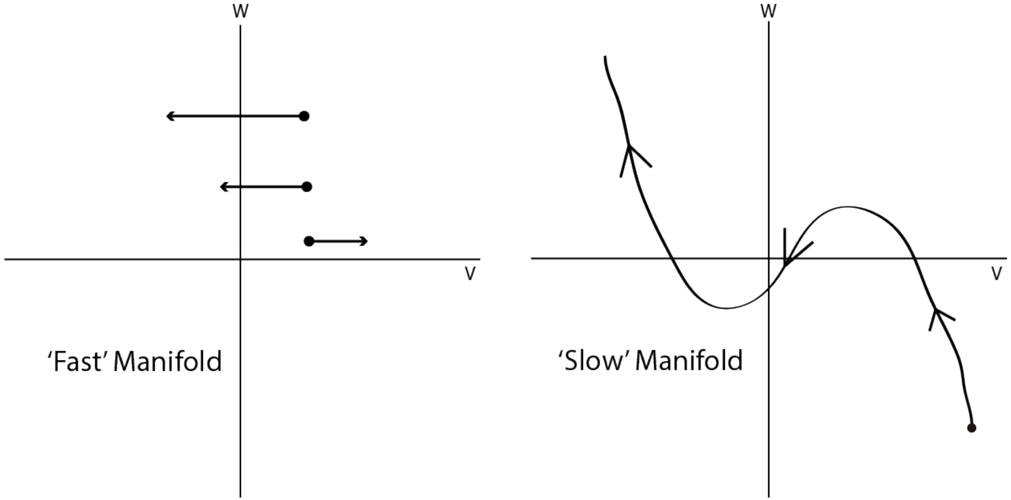
HH



FHN

'Fast' and 'Slow' Time

- **τ** = **ϵt** gives us a short and long time scale: change of variables to get two manifolds

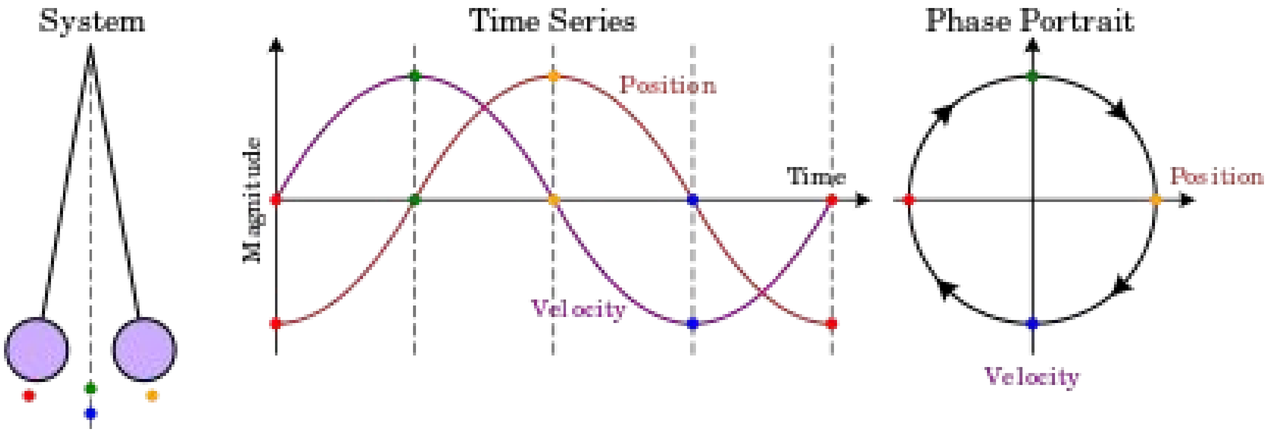


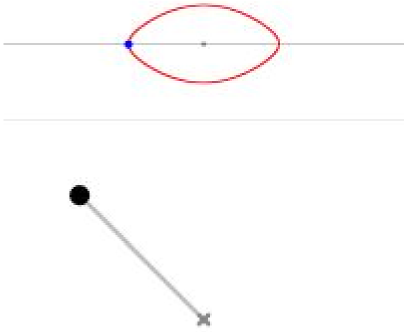
$$\begin{cases} \dot{V} & = V - V^3 - W + I \\ \dot{W} & = 0, \end{cases}$$

$$\begin{cases} 0 & = V - V^3 - W + I \\ \dot{W} & = (V - W). \end{cases}$$

Phase Space

- Plotting all the possible system states:

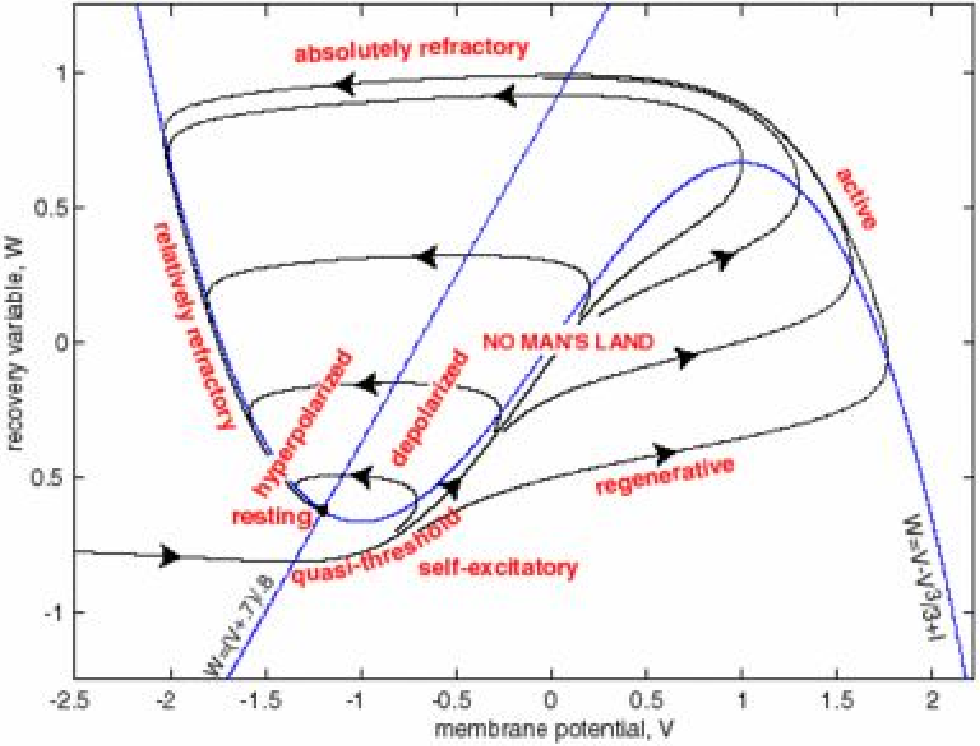




- A powerful tool in complicated systems

The FitzHugh-Nagumo Phase Portrait

- Blue lines are “nullclines,” lines where one state variable isn’t changing
- Notice: similarity to manifolds
- Equilibrium at intersection(s)
- Black lines are time traces of system with different stimuli

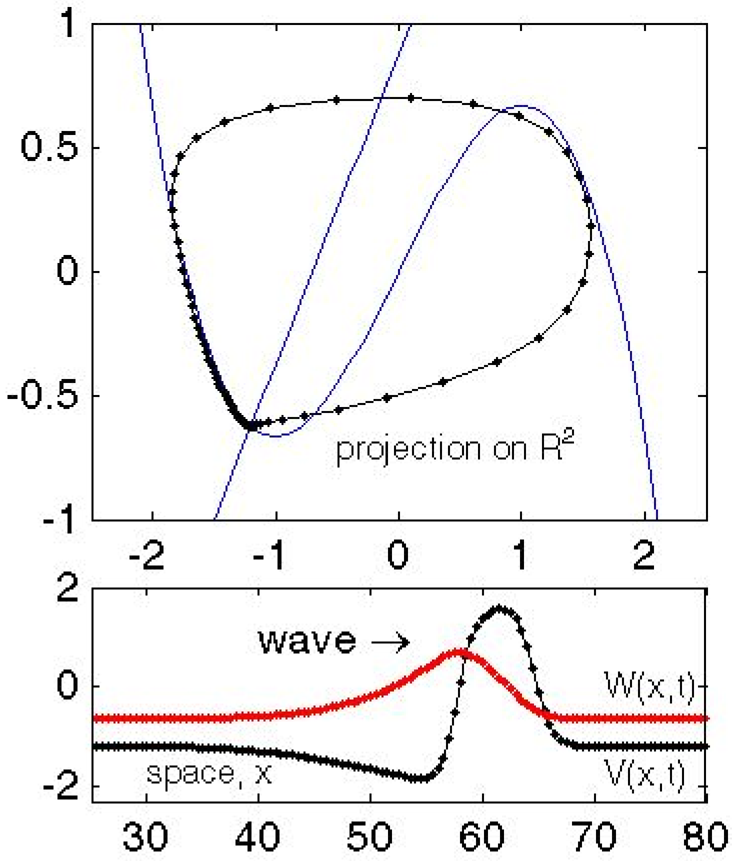


Traveling Waves

- Can our old traveling wave solution work here?

$$\dot{V} = V_{xx} + \underbrace{V - \frac{V^3}{3} - W + I}_{\text{familiar nonlinear bits, f(V)}}$$

- Oscillator + Diffusion Term
  - Biologically: this wave is propagating by ion diffusion so this kind of makes sense…
  - “Reaction-Diffusion Equation”
  - This form gets used all over the place



Traveling Waves: Good and Bad News

If we're hoping for

$$V(t,x) = \Phi(x - ct)$$

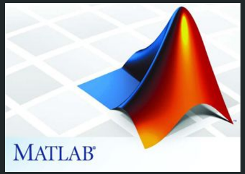
then this must be true

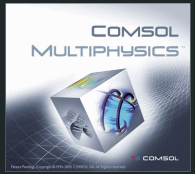
$$-c\Phi' = \Phi'' + f(\Phi)$$

And if *that* solution is bounded, then our traveling wave soln. can be physically reasonable

But myelin = spatially variant capacitance?

We have tools for estimating this grossness:





$$(K(x)V_x)_x = \int V(y)K(x-y)dy$$
$$= V_{n+1} + V_{n-1} - 2V_n$$

Further Analysis + Applications

- Stability Problem
  - If we poke our stable waveform (imperfect axon, imperfect model)-- will the entire signal fall apart or fix itself and successfully reach the next neuron?
- Neural Prostheses
  - Retinal devices, cochlear implants, and neurally-controlled artificial limbs
  - These devices mimic how neurons send signals to the brain and each other
- Networks of Neurons
  - Mathematically interesting connections between how individuals neurons fire & network fn.
  - Epilepsy research

Thank you!

(Shoutout: particularly you, Aaron…)